Journal of Engineering Physics and Thermophysics, Vol. 74, No. 3, 2001

HYDRODYNAMIC INTERACTION AND DEFORMATION OF DROPS

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UDC 532.529:541.182

Hydrodynamic interaction of drops in a linear flow of a viscous incompressible fluid is considered. An analytical solution is suggested for the problem of two and three interacting drops. The forces acting on the drops and the relative velocities of the latter are calculated. The shape of the surface for two interacting drops is found. Comparison with the earlier obtained results is made.

The present work deals with the investigation of hydrodynamic interaction and deformation of drops in the linear field of a viscous incompressible fluid flow. The problem of flow of one liquid around the liquid particles of another one has been the focus of a large number of works. Hydrodynamic interaction of two liquid spheres in a homogeneous flow was studied in [1]. The solution was represented in the form of a series in a bispherical coordinate system. In [2], axial motion of a drop inside a tube was considered with account for its interaction with the walls. In [3], a more general formulation of the problem was considered, with one of the spheres lying inside the other. Asymmetric motion of two liquid spheres was considered in [4]. In [3, 4], a method of numerical solution of the problem is suggested whose procedure gives a good convergence of the series. Hydrodynamic interaction of undeformed spherical drops in a flow with a linear velocity profile was studied in [5]. The solution was also represented as a series of special functions obtained with the use of a bispherical coordinate system. The absence of deformation of the drops in all the cases analyzed leads to a linear formulation of the problem. Deformation of a separate drop in homogeneous and linear flow was considered in numerous works (see, e.g., the review [6]).

In all of the above-mentioned works, the solution of the problems concerning the hydrodynamic interaction of two drops or of a drop with a tube wall was sought by the method of reflections in a bispherical coordinate system, which in principle does not allow one to take into account deformation of drops or consider interaction of three and more liquid particles. Thus, the reflection method does not give a full enough solution of this problem. Therefore, developing a method of finding an analytical solution of this kind of problems is of considerable interest for both different applications and an understanding of the fundamental mechanisms that operate in disperse media with liquid particles; this is very important for investigating the rheological behavior of highly concentrated emulsions by analytical methods.

In [7, 8], we proposed a method of analytical solution of the problem concerning hydrodynamic interaction of a finite number of solid particles in flows whose velocity at infinity is represented as a polynomial of any integral degree. It is shown, in particular, that the solution of the problem concerning hydrodynamic interaction of three particles cannot be reduced in principle to a sum of solutions of the problems of paired interactions of these particles. It is obvious that a similar conclusion can also be drawn for the interaction of liquid particles, thus confirming again the urgency of finding the method of solution of similar problems.

In the present work, the hydrodynamic interrelationship between two drops is studied with account for their deformation as a result of interaction both with the main flow of liquid and among themselves and

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also that of three undeformed spherical drops. It is shown that the method of [7, 8] for analytical solution of the problem of hydrodynamic interaction of solid particles can also be applied in the case of liquid particles.

Let us consider the hydrodynamic interaction of two previously spherical drops A and B of the same volume that corresponds to a sphere of radius d; they are placed into an infinite incompressible liquid with viscosity η. The liquid inside each drop is incompressible and has viscosity η_1. The densities of the liquids outside and inside the drops are identical. The external forces, except for the gravity force, do not act. The dimensions of the liquid particles are small, and the Reynolds number is smaller than unity. The liquid velocity at infinity **U** is a linear function of the coordinates

$$U_i = \Gamma_{ij}\, x_j + \Omega_{ij}\, x_j \,,$$

$$\Gamma_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \quad \Omega_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right).$$

The position occupied by the point of the carrying liquid relative to the selected centers in the drops A and B will be denoted by the vectors $\mathbf{X}_\mathrm{a}$ and $\mathbf{X}_\mathrm{b}$, respectively. For them, the following relation holds:

$$\mathbf{X}_\mathrm{b} = \mathbf{X}_\mathrm{a} - \mathbf{r}\,.$$

Since the Reynolds number is smaller than unity, the equations for the velocity $\mathbf{u}(\mathbf{x})$ and pressure $p(\mathbf{x})$ in the carrying liquid and for the velocity $\mathbf{v}^\mathrm{a}(\mathbf{x})$ and pressure $p^2(\mathbf{x})$ inside the drop A are written in the Stokes approximation:

$$\nabla \mathbf{u} = 0\,, \quad \nabla \mathbf{v}^\mathrm{a} = 0\,, \tag{1}$$

$$\eta \nabla^2 \mathbf{u} = \nabla p\,, \quad \eta_1 \nabla^2 \mathbf{v}^\mathrm{a} = \nabla p^\mathrm{a}\,. \tag{2}$$

The presence of the antisymmetric tensor Ω_{ij} in the expression for the carrying liquid velocity (1) does not influence the distribution of pressure inside a drop and can be taken into account simply by adding the term $\Omega_{ij} X_{aj}$ to the expression for the velocity inside the drop $\mathbf{v}^\mathrm{a}$. Therefore, the boundary conditions on the surface of the drop A can be written as follows:

$$(u_i + U_i\,(\mathrm{A}) - V_i^\mathrm{a} + \Gamma_{ij}\, X_{\mathrm{a}j})\, n_i^\mathrm{a} = 0\,, \quad F_\mathrm{a}\,(\mathbf{X}_\mathrm{a}) = 0\,; \tag{3}$$

$$u_i + U_i\,(\mathrm{A}) + \Gamma_{ij}\, X_{\mathrm{a}j} = V_i^\mathrm{a} + v_i^\mathrm{a}\,, \quad F_\mathrm{a}\,(\mathbf{X}_\mathrm{a}) = 0\,, \tag{4}$$

$$\eta\left(\Gamma_{ij} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j^\mathrm{a}\, \tau_i^\mathrm{a} = \eta_1 \left(\frac{\partial v_i^\mathrm{a}}{\partial x_j} + \frac{\partial v_j^\mathrm{a}}{\partial x_i}\right) n_j^\mathrm{a}\, \tau_i^\mathrm{a}\,, \quad F_\mathrm{a}\,(\mathbf{X}_\mathrm{a}) = 0\,; \tag{5}$$

$$-p + \eta\left(\Gamma_{ij} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j^\mathrm{a}\, n_i^\mathrm{a} = -p^\mathrm{a} + \eta_1 \left(\frac{\partial v_i^\mathrm{a}}{\partial x_j} + \frac{\partial v_j^\mathrm{a}}{\partial x_i}\right) n_j^\mathrm{a}\, n_i^\mathrm{a} + \alpha\,(K_1 + K_2)\,, \quad F_\mathrm{a}\,(\mathbf{X}_\mathrm{a}) = 0\,. \tag{6}$$

Far from the drop, the perturbations decay:

$$u_i \to 0\,, \quad |\mathbf{X}_\mathrm{a}| \to \infty\,, \quad p \to 0\,, \quad |\mathbf{X}_\mathrm{a}| \to \infty\,. \tag{7}$$

Similar equations and boundary conditions can be written for the velocity $\mathbf{v}^b(\mathbf{x})$ and pressure $p^b(\mathbf{x})$ inside the drop B. The equation of the surface of each drop is an unknown function determined by solving the problem. Since the shape of the surface changes with time, it would seem that nonstationary equations of a slow flow must be taken into account. But the linear velocity of this surface has the order of the small velocity of motion of a particle relative to the liquid, while the nonstationary terms account for the order of convective inertia terms that are neglected. Moreover, the nonstationarity is induced because of the motion of particles relative to one another. However, if the distance between the surfaces of the particles has the order of the size of the particles, then one may also neglect the nonstationary terms in the equations of motion.

The method of solving the foregoing problem is similar to that suggested in [7, 8] for solid particles. With account for the fact that the liquid is incompressible, the divergence of both sides of Eq. (2) yields the equation for pressure:

$$\nabla^2 p = 0 \,.$$

Solving it with account for the condition at infinity (7) and substituting it into (2), we obtain the equation for the velocity. The solution of the equations for the liquid outside drops that satisfies the condition at infinity (7) can be written in the same form as for solid particles [8]:

$$p = H_i^a L_i^a + H_i^b L_i^b + F_{ij}^a L_{ij}^a + F_{ij}^b L_{ij}^b + G_{ijk}^a L_{ijk}^a + G_{ijk}^b L_{ijk}^b + \dots \,, \tag{8}$$

$$\eta u_i = -\frac{2}{3}(H_i^a L_0^a + H_i^b L_0^b) - \frac{3}{5}(F_{ij}^a L_j^a + F_{ij}^b L_j^b) - \frac{4}{7}(G_{ijk}^a L_{jk}^a + G_{ijk}^b L_{jk}^b) - \frac{1}{6}(H_j^a L_{ij}^a X_a^2 + H_j^b L_{ij}^b X_b^2) -$$

$$- \frac{1}{10}(F_{jk}^a L_{ijk}^a X_a^2 + F_{jk}^b L_{ijk}^b X_b^2) - \frac{1}{14}(G_{jkl}^a L_{ijkl}^a X_a^2 + G_{jkl}^b L_{ijkl}^b X_b^2) - \dots \tag{9}$$

Here $L_{ij\dots k}$ is the multipole calculated according to the following rule:

$$L_{ij\dots s} = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\left(\dots\left(\frac{\partial}{\partial x_s}\left(\frac{1}{X}\right)\right)\right)\right),$$

where the quantity X is the distance from the center of a drop to the point in the liquid where the value for pressure is taken. For the case of two drops, the solution must depend on the distances to the two centers. Therefore, the expressions for pressure and velocity must contain multipoles of two types: with partial derivatives of the function $1/X_a$ and of the function $1/X_b$.

Inside each drop the functions for pressure and velocity must not contain singularities at $\mathbf{X}_a = 0$ and at $\mathbf{X}_b = 0$. This solution can be obtained using the same procedure as that for solid particles [7, 8]. For the drop A it is written as follows:

$$p^a = M_i^a L_i^a X_a^3 + Q_{ij}^a L_{ij}^a X_a^5 + R_{ijk}^a L_{ijk}^a X_a^7 + D_{ijkl}^a L_{ijkl}^a X_a^9 + \dots \,, \tag{10}$$

$$\eta_1 v_i^a = E_i^a + M_s^a\left[\frac{1}{2}L_s^a X_a^3 X_{ai} + \frac{1}{5}L_{is}^a X_a^5\right] + Q_{st}^a\left[\frac{1}{2}L_{st}^a X_a^5 X_{ai} + \frac{5}{42}L_{sti}^a X_a^7\right] + R_{stq}^a\left[\frac{1}{2}L_{stq}^a X_a^7 X_{ai} + \frac{1}{12}L_{istq}^a X_a^9\right] +$$

$$+ D_{stqr}^a\left[\frac{1}{2}L_{stqr}^a X_a^9 X_{ai} + \frac{7}{110}L_{istqr}^a X_a^{11}\right] + \dots + K_{ij}^a L_j^a X_a^3 + N_{ijk}^a L_{jk}^a X_a^5 + S_{ijkl}^a L_{jkl}^a X_a^7 + \dots +$$

$$+ \nabla_i\,[T_j^a L_j^a X_a^3 + P_{jk}^a L_{jk}^a X_a^5 + W_{jkl}^a L_{jkl}^a X_a^7 + \dots] \,. \tag{11}$$

The second and third groups of terms in the expression for the velocity of the liquid inside a particle are the solutions of the Laplace and continuity equations; moreover, the second group involves the conditions for the tensor coefficients:

$$N^{\mathrm{a}}_{jj} = 0\ , \quad S^{\mathrm{a}}_{ijj} = S^{\mathrm{a}}_{jji} = S^{\mathrm{a}}_{jij} = 0\ , \ ...$$

Expressions for the velocity $\mathbf{v}^{\mathrm{b}}(\mathbf{x})$ and pressure $p^{\mathrm{b}}(\mathbf{x})$ inside the drop B are written in a similar way.

Since the shape of the surface of each drop depends on the position of the drops relative to one another r_j, the distance between them d/r, the velocity of drops relative to the main flow $\mathbf{V}^{\mathrm{a}} - \mathbf{U}(\mathrm{A})$, $\mathbf{V}^{\mathrm{b}} - \mathbf{U}(\mathrm{B})$, the tensor $\mathbf{\Gamma}_{ij}$, and the surface tension coefficient α, the unknown tensor coefficients contained in the expressions for the velocity and pressure must depend on these quantities and, in contrast to the case of spherical particles, they must be nonlinear with respect to $\mathbf{V}^{\mathrm{a}} - \mathbf{U}(\mathrm{A})$, $\mathbf{V}^{\mathrm{b}} - \mathbf{U}(\mathrm{B})$, and $\mathbf{\Gamma}_{ij}$. Using the tensors $\mathbf{V}^{\mathrm{a}} - \mathbf{U}(\mathrm{A})$, $\mathbf{V}^{\mathrm{b}} - \mathbf{U}(\mathrm{B})$, $\mathbf{\Gamma}_{ij}$, r_j, and δ_{ij}, we can construct a tensor of any rank, nonlinear in $\mathbf{V}^{\mathrm{a}} - \mathbf{U}(\mathrm{A})$, $\mathbf{V}^{\mathrm{b}} - \mathbf{U}(\mathrm{B})$, and $\mathbf{\Gamma}_{ij}$ and containing only the scalar function of the parameters d/r and α. Taking all possible combinations of the tensors $\mathbf{V}^{\mathrm{a}} - \mathbf{U}(\mathrm{A})$, $\mathbf{V}^{\mathrm{b}} - \mathbf{U}(\mathrm{B})$, $\mathbf{\Gamma}_{ij}$, r_k, and δ_{en} that are nonlinear in the quantities $\mathbf{V}^{\mathrm{a}} - \mathbf{U}(\mathrm{A})$, $\mathbf{V}^{\mathrm{b}} - \mathbf{U}(\mathrm{B})$, and $\mathbf{\Gamma}_{ij}$, it is possible to write expressions for the unknown tensor coefficients in the expressions for the velocity and pressure both inside and outside drops. For example, the tensor coefficient H^{a}_i can be represented as

$$H^{\mathrm{a}}_i = \eta\, \Big\{ (V^{\mathrm{a}}_i - U_i\,(\mathrm{A}))\, H\mathrm{A} + (V^{\mathrm{a}}_i - U_i\,(\mathrm{A}))\,(V^{\mathrm{a}}_j - U_j\,(\mathrm{A}))\, r_j\, H\mathrm{B} + (V^{\mathrm{a}}_j - U_j\,(\mathrm{A}))\,(V^{\mathrm{a}}_j - U_j\,(\mathrm{A}))\, r_i\, H\mathrm{C} +$$

$$+ \Gamma_{ij}\,(V^{\mathrm{a}}_j - U_j\,(\mathrm{A}))\, H\mathrm{D} + \Gamma_{ij}\, r_j\, E\mathrm{A} + \Gamma_{jk}\, r_j\, r_k\, r_i E\mathrm{B} + \Gamma_{ij}\,\Gamma_{jk}\, r_k\, F\mathrm{A} + \Gamma_{jk}\,\Gamma_{jk}\, r_i\, F\mathrm{B} + ... \Big\}\,.$$

The unknown scalar functions (EA, EB, etc.) in these expressions depend on the shape of the surface, the liquid viscosity outside and inside drops, and the distance between their centers and surface tension and is determined from the boundary conditions.

To calculate the unknown scalar functions contained in the tensor coefficients, we consider boundary condition (6). Using the well-known relations for the main curvatures of the surface, we obtain the following expression for the drop A (the sum of the main curvatures can be written similarly also for the drop B):

$$K_1 + K_2 = -\frac{1}{|\nabla F_{\mathrm{a}}|}\, \nabla\, (\nabla F_{\mathrm{a}})\,.$$

Assuming the surface tension coefficient to be rather high, so that the parameters $\eta\Gamma d/\alpha$ and $\eta U/\alpha$ are small (U and Γ are the absolute values of the relative velocity of particles and of the velocity gradient of the main flow, respectively), the surface equation for the particle A will be sought in the form of the series

$$F_{\mathrm{a}}\,(\mathbf{X}_{\mathrm{a}}) = X_{\mathrm{a}} - d - \eta\,(V^{\mathrm{a}}_i - U_i\,(\mathrm{A}))\,\frac{X_{\mathrm{a}i}}{d\alpha}\, f_1 + \eta\,(V^{\mathrm{a}}_i - U_i\,(\mathrm{A}))\,(V^{\mathrm{a}}_j - U_j\,(\mathrm{A}))\,\frac{X_{\mathrm{a}i}\, X_{\mathrm{a}j}}{(d\alpha)^2}\, f_2 + ... +$$

$$+ \eta\Gamma_{ij}\,\frac{X_{\mathrm{a}i}\, X_{\mathrm{a}j}}{d\alpha}\, g_1 + \eta\Gamma_{ij}\,\Gamma_{kn}\, X_{\mathrm{a}i}\, X_{\mathrm{a}j}\,\frac{X_{\mathrm{a}k}\, X_{\mathrm{a}n}}{(d\alpha)^2}\, g_2 + ... + \eta\,(V^{\mathrm{a}}_i - U_i\,(\mathrm{A}))\,\Gamma_{jk}\, X_{\mathrm{a}i}\, X_{\mathrm{a}j}\,\frac{X_{\mathrm{a}k}}{(d\alpha)^2}\, h_1 + ... + ... \qquad (12)$$

Here f, g, and h with the corresponding subscripts denote the scalar functions that depend on the parameter d/r. For small values of the parameter $\varepsilon = d/r$, calculations of the unknown scalar functions in the expressions for the velocity and pressure are similar to those for the case of solid particles [8] with the only difference being that the number of unknowns increases considerably, and this leads to the familiar mathematical differences. Calculations can be made with any accuracy using small parameters. The results of the calculations of scalar functions for the linear quantities that are linear in velocity and velocity gradient are given below.

Substituting the expressions for the liquid velocity outside **u** (9) and inside $\mathbf{v}^a$ (11) the drop A, with account for the form of the tensor coefficients in them, into boundary conditions (3)–(5) and assuming that the order of the quantities $\eta\Gamma d/\alpha$ and $\eta U/\alpha$ is the same, we obtain the values for the unknown functions in the form of a series in the parameter ε. For interacting spherical drops in a homogeneous flow, we obtain the well-known solution of Hadamard–Rybchinskii (the case $\varepsilon = 0$ and $\eta U/\alpha = 0$) in the notation introduced above. It has the form

$$\eta u_i = -\frac{2}{3} H_i^a L_0^a - \frac{1}{6} H_j^a L_{ij}^a X_a^2 - \frac{4}{7} G_{ijk}^a L_{jk}^a , \quad \eta_1 v_i^a = \frac{1}{2} M_k^a L_k^a X_a^3 X_{ai} + \frac{1}{5} M_j^a L_{ij}^a X_a^5 + E_i^a ,$$

$$H_i^a = \eta\, (V_i^a - U_i\,(\mathrm{A}))\, H , \quad G_{ijk}^a = \eta\, [(V_i^a - U_i\,(\mathrm{A}))\, \delta_{jk} + (V_j^a - U_j\,(\mathrm{A}))\, \delta_{ik} + (V_k^a - U_k\,(\mathrm{A}))\, \delta_{ji}]\, G ,$$

$$E_i^a = \eta_1\, (V_i^a - U_i\,(\mathrm{A}))\, E^a , \quad M_i^a = \eta_1\, (V_i^a - U_i\,(\mathrm{A}))\, M^a .$$

The values of the coefficients are

$$H = \frac{d}{2}\lambda , \quad G = -\frac{7d^3}{32}\nu , \quad M^a = -\frac{5}{d^2}\nu , \quad E^a = -\frac{\nu}{2} , \quad \lambda = \frac{3\eta_1 + 2\eta}{\eta_1 + \eta} , \quad \nu = \frac{\eta}{\eta_1 + \eta} .$$

For moving drops it is necessary to take into account the fact that both the shape of the particles and their linear velocity relative to the liquid change. Moreover, just as for solid particles, it is necessary to take into account separately changes in the linear velocity both along the vector **r** and normal to it. This means that in constructing tensor coefficients one should take combinations not with the linear velocity of drops but with its components. The values of these velocities are calculated from the condition of the equality to zero of the forces that act on each particle with account for the deformation of its surface. In the present work, calculations were performed up to the value of the order of ε^3 (on the assumption that the order of the quantities $\eta\Gamma d/d$ and $\eta U/\alpha$ is much smaller than that of ε^3). The results obtained coincide in the limit $\eta\Gamma d/\alpha \to 0$ with the calculations of [5] for undeformed spherical liquid particles.

With account for the dependence of the drop velocity $\mathbf{V}^a - \mathbf{U}(\mathrm{A})$ on the gradient of the velocity of the main flow Γ_{ij}, the equation of the surface of the drop A (12) can be represented in the form

$$X_a = d + \Phi_1 \frac{\eta}{\alpha} \Gamma_{jk}\, r_j\, r_k\, (r_l\, X_{al}) + \Phi_2 \frac{\eta}{\alpha} \Gamma_{jk}\, r_j\, r_k + \Phi_3 \frac{\eta}{\alpha} \Gamma_{jk}\, X_{aj}\, X_{ak} + \Phi_4 \frac{\eta}{\alpha} \Gamma_{jk}\, r_j\, X_{ak}\, (r_l\, X_{al}) +$$

$$+ \Phi_5 \frac{\eta}{\alpha} \Gamma_{jk}\, r_j\, X_{ak}\, (r_l\, X_{al})^2 + \Psi_1 \frac{\eta_1}{\alpha} \Gamma_{jk}\, r_j\, r_k\, (r_l\, X_{al}) + \Psi_2 \frac{\eta_1}{\alpha} \Gamma_{jk}\, r_j\, r_k + \Psi_3 \frac{\eta_1}{\alpha} \Gamma_{jk}\, X_{aj}\, X_{ak} +$$

$$+ \Psi_4 \frac{\eta_1}{\alpha} \Gamma_{jk}\, r_j\, X_{ak}\, (r_l\, X_{al}) + \Psi_5 \frac{\eta_1}{\alpha} \Gamma_{jk}\, r_j\, X_{ak}\, (r_l\, X_{al})^2 . \tag{13}$$

Substituting this expression into boundary condition (6) with account for the pressure and velocity outside, (8)–(9), and inside, (10)–(11), a drop and the found values for the tensor coefficients in them, we obtain the following values of the parameters:

$$\Phi_1 = \frac{3}{4r^2}\mu\varepsilon^3 \left[1 + \frac{1}{2}\lambda\varepsilon + \frac{1}{4}\lambda^2\varepsilon^2\right], \quad \Psi_1 = 2\nu\Phi_1 , \quad \Phi_3 = -2\nu , \quad \Psi_3 = 19\nu , \quad \Phi_4 = \frac{5}{r^2}\mu\kappa\varepsilon^3 ,$$

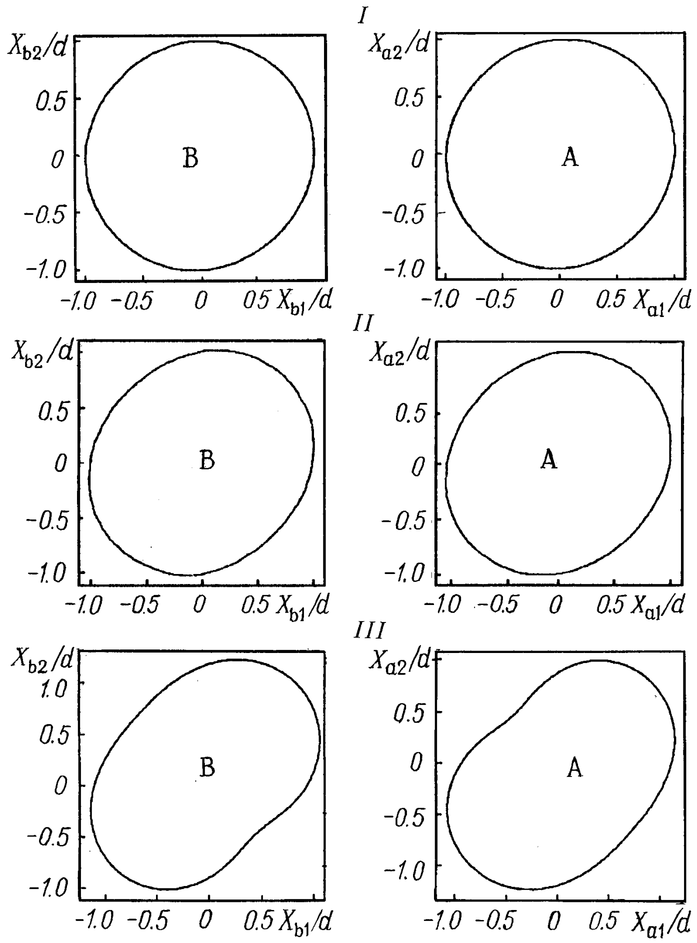

Fig. 1. Deformation of the drops A and B: I) d/r = 0.03; II) 0.1; III) 0.3.

$$\Psi_4 = 0\,,\quad \Phi_5 = -\frac{79}{6r^4}\mu\xi\varepsilon^3\,,\quad \Psi_5 = -\frac{\Phi_5}{4}\,,\quad \mu = \frac{5\eta_1 + 2\eta}{\eta_1 + \eta}\,,\quad \kappa = \frac{\eta}{5\eta_1 + 8\eta}\,,\quad \xi = \frac{\eta}{5\eta_1 + 4\eta}\,.$$

The found values of the coefficients correspond to the deformation of the drop A. Similar calculations can be easily made for the drop B. If the beginning of reckoning of the coordinate system is placed at the center of the drop B, then, taking into account that the vector **r** will reverse its direction, we obtain the same values of the coefficients in the expression for the shape of the surface but opposite signs before the terms that contain the odd degrees of the components of the vector **r**. Figure 1 shows the shapes of main cross sections of the surfaces of the drops A and B for linear flow of simple shear (only the components $\Gamma_{12} = \Gamma_{21}$ in the tensor Γ_{ij} differ from zero) for the relative position of their centers corresponding to the vector $\mathbf{r} = (-r\sqrt{2}/2, r\sqrt{2}/2, 0)$. It is seen from the calculations that at large distances between the drops and small parameters $\eta U/\alpha$ and $\eta\Gamma d/\alpha$ the shape of the drops is close to an ellipsoidal one. With decrease in the distance between drops, the shape of the surface becomes lens-like.

The determined distributions of the pressure and velocity make it possible to calculate the forces acting from the side of the liquid on the drops. In the approximation considered, the drop A experiences the action of the force **F** equal to

$$F_i = d\pi\eta\lambda \left\{ \Gamma_{ij}\, r_j \frac{\sigma}{\lambda}\varepsilon^5 + \mu\Gamma_{jk}\, r_j\, r_k\, r_i \left[\varepsilon^3 + \frac{\lambda}{2}\varepsilon^4\right] - U_i^{a\|}\left[2 + \lambda\frac{3}{2}\varepsilon^2\right] + U_i^{b\|}\lambda\varepsilon - U_i^{a\perp}\left[2 + \frac{3}{8}\lambda\varepsilon^2\right] + U_i^{b\perp}\frac{\lambda\varepsilon}{2} \right\}. \quad (14)$$

Here $\sigma = 8\eta_1(\eta + 2\eta_1)/(\eta + \eta_1)^2$, $U_i^{a\parallel}$, $U_i^{a\perp}$, $U_i^{b\parallel}$, and $U_i^{b\perp}$ are the components of the vectors $\mathbf{V}^a - \mathbf{U}(A)$ and $\mathbf{V}^b - \mathbf{U}(B)$ along the vector $\mathbf{r}$ and normal to it, respectively. The force acting on the drop B is obtained from expression (14) by interchanging the superscripts a and b and replacing the vector $\mathbf{r}$ by $-\mathbf{r}$. The expressions obtained for the forces were used in finding the dependence of the velocity of drops $\mathbf{V}^a - \mathbf{U}(A)$ and $\mathbf{V}^b - \mathbf{U}(B)$ on the gradient of the velocity of the main flow Γ_{ij} in equality (13) for the deformation of the surface of the drop A and in a similar equality for the drop B. This dependence follows from the requirement that the forces acting on each drop be equal to zero. This very condition for forces yields expressions for the velocity of the drop B relative to the drop A:

$$V_i^b - V_i^a = \Omega_{ij} r_j + \Gamma_{ij} r_j (1 - J) + \Gamma_{jk} \frac{r_j r_k r_i}{r^2} (J - I)$$

with the coefficients

$$I = \mu\varepsilon^3 - \frac{3\sigma}{2\lambda}\varepsilon^5 , \quad J = \frac{\sigma}{\lambda}\varepsilon^5 .$$

The values of the coefficients I and J coincide with the similar coefficients of [5].

The method used in the present work allows one, just as in the case of solid particles [8], to consider the interaction of n drops in the flows whose velocity at infinity is represented as a polynomial of integral degree in the coordinates. We consider the interaction of three drops A, B, and C. The distribution of the pressure and velocity inside the drops is sought in the same form as for the two, and outside the drops it is sought in the same form as for three solid particles [8]:

$$p = H_i^a L_i^a + H_i^c L_i^c + H_i^b L_i^b + F_{ij}^a L_{ij}^a + F_{ij}^c L_{ij}^c + F_{ij}^b L_{ij}^b + \ldots ,$$

$$\eta u_i = -\frac{2}{3} H_i^a L_0^a - \frac{3}{5} F_{ij}^a L_j^a - \frac{2}{3} H_i^c L_0^c - \frac{3}{5} F_{ij}^c L_j^c - \frac{2}{3} H_i^b L_0^b - \frac{3}{5} F_{ij}^b L_j^b - \ldots .$$

The tensor quantities in the expressions for the velocity and pressure outside and inside the drops are represented in the form of the same combinations of the corresponding tensors as in the problem of two drops with the only difference being that the number of possible combinations increases. This is because the relative position of the centers of the three spheres is determined by assigning not one of the radius-vectors, as for two spheres, but two radius-vectors. As these vectors we may select any two radii-vectors out of the three that connect the centers. Correspondingly, for the problem of interaction of four or more drops it is necessary to use any three radii-vectors, not lying in one plane, in order to prescribe their position. The number of possible combinations for tensor quantities here increases. Correspondingly, there is an increase in the number of scalar functions entering into these combinations and the number of algebraic equations needed for their determination. This presents the main difficulty in the method of solving the problem of hydrodynamic interaction of three or more drops. In the present work, the solution of the problem of hydrodynamic interaction of three spheres of the same radius has been found with an accuracy of up to the third order of smallness in the parameter ε. Just as for solid particles, it is impossible to separate the contribution of the interactions, for example, of the liquid particles B and C with the particle A. We can only calculate the total contribution of the interaction of the particle A with the other two. This is explained by the fact that it is impossible to separate the contribution of each particle to the boundary conditions. However, knowledge of the total result of the interaction is quite sufficient for calculation of the forces and moments acting on each liquid particle from the side of two other. Thus, for the particle A we obtain the following expression for the force acting from the side of the liquid as a result of the interaction with the other two:

$$
\begin{aligned}
F_i = d\pi\eta\lambda\mu \Bigg\{ & \Gamma_{jk} \frac{r_j^{\mathrm{a}} r_k^{\mathrm{a}} r_i^{\mathrm{a}}}{r_{\mathrm{a}}^2} \left[\left(1 + \frac{r_{\mathrm{a}}^5}{r_{\mathrm{c}}^5} \right) \varepsilon^3 + \frac{\lambda}{2} \left(1 + \frac{r_{\mathrm{a}}^6}{r_{\mathrm{c}}^6} \right) \varepsilon^4 \right] + \\
& + \Gamma_{jk} \frac{r_j^{\mathrm{b}} r_k^{\mathrm{b}} r_i^{\mathrm{b}}}{r_{\mathrm{b}}^2} \left[\frac{r_{\mathrm{a}}^3 r_{\mathrm{b}}^2}{r_{\mathrm{c}}^5} \varepsilon^3 + \frac{\lambda}{4} \left(\frac{r_{\mathrm{a}}^3}{r_{\mathrm{b}}^3} \left(-1 + \frac{r_{\mathrm{a}}}{r_{\mathrm{c}}} + \frac{r_{\mathrm{a}} r_{\mathrm{b}}^2}{r_{\mathrm{c}}^3} + \frac{r_{\mathrm{a}}^2 r_{\mathrm{b}}}{r_{\mathrm{c}}^3} e_s^{\mathrm{a}} e_s^{\mathrm{b}} \right) + 2 \frac{r_{\mathrm{a}}^4 r_{\mathrm{b}}^2}{r_{\mathrm{c}}^6} \right) \varepsilon^4 \right] + \\
& + \Gamma_{jk} \frac{r_j^{\mathrm{a}} r_k^{\mathrm{b}} r_i^{\mathrm{a}}}{r_{\mathrm{a}}^2} \left[2 \frac{r_{\mathrm{a}}^5}{r_{\mathrm{c}}^5} \varepsilon^3 + 2 \frac{r_{\mathrm{a}}^6}{r_{\mathrm{c}}^6} \lambda \varepsilon^4 \right] + \Gamma_{jk} \frac{r_j^{\mathrm{b}} r_k^{\mathrm{b}} r_i^{\mathrm{a}}}{r_{\mathrm{b}}^2} \left[\frac{r_{\mathrm{a}}^3 r_{\mathrm{b}}^2}{r_{\mathrm{c}}^5} \varepsilon^3 + \frac{\lambda}{4} \left(\frac{r_{\mathrm{a}}^5}{r_{\mathrm{b}}^5} \left(- \frac{r_{\mathrm{b}}}{r_{\mathrm{a}}} e_s^{\mathrm{a}} e_s^{\mathrm{b}} + \right.\right.\right. \\
& \left.\left.\left. + \frac{r_{\mathrm{a}} r_{\mathrm{b}}^2}{r_{\mathrm{c}}^3} + \frac{r_{\mathrm{a}}^2 r_{\mathrm{b}}}{r_{\mathrm{c}}^3} e_s^{\mathrm{a}} e_s^{\mathrm{b}} \right) + 2 \frac{r_{\mathrm{a}}^6}{r_{\mathrm{c}}^6} \right) \varepsilon^4 \right] + \Gamma_{jk} \frac{r_j^{\mathrm{a}} r_k^{\mathrm{b}} r_i^{\mathrm{b}}}{r_{\mathrm{b}}^2} \left[2 \frac{r_{\mathrm{a}}^3 r_{\mathrm{b}}^2}{r_{\mathrm{c}}^5} \varepsilon^3 + \frac{r_{\mathrm{a}}^6}{r_{\mathrm{c}}^6} \lambda \varepsilon^4 \right] + \\
& + \Gamma_{jk} \frac{r_j^{\mathrm{a}} r_k^{\mathrm{a}} r_i^{\mathrm{b}}}{r_{\mathrm{b}}^2} \left[\frac{r_{\mathrm{a}}^3 r_{\mathrm{b}}^2}{r_{\mathrm{c}}^5} \varepsilon^3 + \frac{\lambda}{2} \frac{r_{\mathrm{a}}^6}{r_{\mathrm{c}}^6} \varepsilon^4 \right] \Bigg\} - 2 d\pi\eta\lambda \left[U_i^{\mathrm{a}} + \frac{\lambda}{4} \left(2 U_s^{\mathrm{b}} e_s^{\mathrm{a}} + U_s^{\mathrm{b}} e_s^{\mathrm{c}} \frac{r_s^{\mathrm{a}} r_s^{\mathrm{c}}}{r_{\mathrm{a}} r_{\mathrm{c}}} + U_s^{\mathrm{c}} e_s^{\mathrm{a}} \frac{r_{\mathrm{a}}}{r_{\mathrm{c}}} \right) e_i^{\mathrm{a}} \varepsilon + \right. \\
& \left. + \frac{\lambda}{4} \left(U_s^{\mathrm{b}} r_s^{\mathrm{c}} + U_s^{\mathrm{c}} r_s^{\mathrm{a}} \frac{r_s^{\mathrm{a}} r_s^{\mathrm{c}}}{r_{\mathrm{c}}^2} + 2 U_s^{\mathrm{c}} e_s^{\mathrm{c}} \frac{r_{\mathrm{a}}}{r_{\mathrm{c}}} \right) e_i^{\mathrm{c}} \varepsilon + \frac{\lambda}{4} \left(U_i^{\mathrm{b}\perp} + U_i^{\mathrm{c}\perp} \frac{r_{\mathrm{a}}}{r_{\mathrm{c}}} \right) \varepsilon \right].
\end{aligned}
\tag{15}
$$

Here, the vectors $\mathbf{r}_{\mathrm{a}}$, $\mathbf{r}_{\mathrm{c}}$, and $\mathbf{r}_{\mathrm{b}}$ denote the radii-vectors between the drops A and B, A and C, and B and C, respectively, and the unit vectors $\mathbf{e}_{\mathrm{a}} = \mathbf{r}_{\mathrm{a}}/r_{\mathrm{a}}$, $\mathbf{e}_{\mathrm{b}} = \mathbf{r}_{\mathrm{b}}/r_{\mathrm{b}}$, and $\mathbf{e}_{\mathrm{c}} = \mathbf{r}_{\mathrm{c}}/r_{\mathrm{c}}$ are introduced. The vector $\mathbf{U}^{\perp}$ denotes the velocity component perpendicular to the plane in which the drops are located. The expressions for the forces acting on the drops B and C are calculated analogously. Being cumbersome, they are not given here. Comparison of expressions (14) and (15) shows that the force acting on the drop in the case of three interacting liquid particles is not reduced to the sum of the forces acting in paired interactions.

This work was carried out with financial support from Norwegian Research Council, project 107214/410. The author is grateful to Prof. A. Bertelson and Prof. E. Palm from the Oslo University for their interest in the work and for useful remarks.

NOTATION

d, radius of a spherical drop; $\mathbf{n}^{\mathrm{a}}$ and τ^{a}, unit vectors of the normal and tangent to the surface of the drop A as assigned by the equation $F_{\mathrm{a}}(\mathbf{X}_{\mathrm{a}}) = 0$; α, surface tension coefficient; $K_1 + K_2$, the sum of the main curvatures of the surface of the drop A; $\mathbf{U}(\mathrm{A})$, velocity of the nonperturbed flow of liquid at the center of the drop A; $\mathbf{U}(\mathrm{B})$, same at the center of the drop B; η_1, viscosity of liquid in a drop; η, viscosity of the carrying liquid; p^{a}, pressure in the drop A; p^{b}, same in the drop B; $\mathbf{v}^{\mathrm{a}}$, velocity of liquid in the drop A; $\mathbf{v}^{\mathrm{b}}$, same in the drop B; $\mathbf{V}^{\mathrm{a}}$, velocity of the center of mass of the drop A as a whole; $\mathbf{V}^{\mathrm{b}}$, same of the drop B; p, pressure in the carrying liquid; $\mathbf{u}$, velocity of the carrying liquid; Γ_{ij} and Ω_{ij}, symmetric and antisymmetric velocity tensors; x_i, coordinates of the point of liquid; $\mathbf{r}$, vector that connects the centers of the particles; r, absolute value of the vector $\mathbf{r}$; $\mathbf{F}$, the force acting on the drop A; H_i^{a}, F_{ij}^{a}, G_{ijk}^{a}, H_i^{b}, F_{ij}^{b}, H_i^{c}, and F_{ij}^{a}, tensor coefficients for the velocity and pressure in the carrying liquid; M_i^{a}, Q_{ij}^{a}, R_{ijk}^{a}, D_{ijkl}^{a}, K_{ij}^{a}, N_{ijk}^{a}, S_{ijkl}^{a}, T_{ij}^{a}, P_{ij}^{a}, and W_{ijk}^{a}, same in the drop A; HA, HB, HC, HD, EA, EB, FA, FB, scalar coefficients determined from boundary conditions; $L_{ij\ldots s}^{\mathrm{a}}$., multipole of the function $1/X_{\mathrm{a}}$; $L_{ij\ldots s}^{\mathrm{b}}$, same of the function $1/X_{\mathrm{c}}$; X_{a}, the distance from the point in the liquid to the center of the drop A; X_{b}, same to the center of the drop B; X_{c}, same to the center of the drop C; $X_{\mathrm{a}i}$, coordinates of the vector $\mathbf{X}_{\mathrm{a}}$; λ, μ, ν, κ, ξ, and σ, parameters that characterize the interaction of a drop with the flow; $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2$, $\boldsymbol{\Phi}_3$, $\boldsymbol{\Phi}_4$, $\boldsymbol{\Phi}_5$, $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$, and $\boldsymbol{\Psi}_5$, coefficients in the expression for the shape of the surface of the drop A; $\mathbf{U}^a$, $\mathbf{U}^b$, and $\mathbf{U}^c$, relative velocities of the drops A, B, and C, respectively; $\mathbf{U}^{a\|}$, $\mathbf{U}^{a\perp}$, $\mathbf{U}^{b\|}$,$\mathbf{U}^{b\perp}$, velocities of the drops A and B along the vector $\mathbf{r}$ and normal to it; $\mathbf{r}_a$, $\mathbf{r}_c$, and $\mathbf{r}_b$, radii-vectors between the drops A and B, A and C, and B and C, respectively; $\mathbf{e}_a$, $\mathbf{e}_b$, $\mathbf{e}_c$, unit vectors along the vectors $\mathbf{r}_a$, $\mathbf{r}_b$, and $\mathbf{r}_c$. The subscripts *i*, *j*, *k*, ..., *s* denote the components of vectors and tensors.

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